St George Girls High School

Year 11 – Higher School Certificate Course

Assessment Task 1

December 2007

Mathematics Extension 1

Time Allowed: 75 Minutes
(plus 5 minutes reading time)

Instructions to Candidates

1. Write using black or blue pen.
2. Attempt all questions.
3. Start each question on a new page.
4. Show all necessary working.
5. Marks for each question are shown in the right column.
6. Complete cover sheet clearly showing:
   - your name
   - your mathematics class and teacher.

Marks

<u>Question 1</u> – (12 marks) – Start a New Page

a) Find the centre and radius of the circle with equation given by:

$x^2 - 8x + y^2 + 14y + 1 = 0$ 3

b) (i) Derive the equation of the normal to the parabola $x^2 = 4ay$ at the point $(2at, at^2)$ 3

(ii) Find the coordinates of the point N where this normal meets the axis of the parabola.

c) Given that $f(x) = 3^x$ complete the table. 3

x	0	1	2	3	4
3^x					

With these five function values estimate $\int_0^4 3^x \, dx$ using Simpson's Rule.

d) (i) Differentiate $\sqrt{1+x^4}$ 3

(ii) Hence find $\int \frac{x^3 \, dx}{\sqrt{1+x^4}}$

Question 2 – (12 marks) – Start a new page **Marks**

a) For the function $y = 4x^3 - x^4$ 8

(i) Show that there are 2 stationary points.

(ii) Find the nature of the stationary points.

(iii) Find the point(s) on the curve where the concavity changes.

(iv) Sketch the curve showing intercepts and the above features.

b) Find the equation of the parabola with axis parallel to the y-axis and passing through the points $(0, -2)$, $(1, 0)$ and $(3, -8)$ 4

Question 3 – (12 marks) – Start a new page **Marks**

a) A parabola has its vertex at the point $(2, 1)$ and focus at the point $(2, 3)$. 3

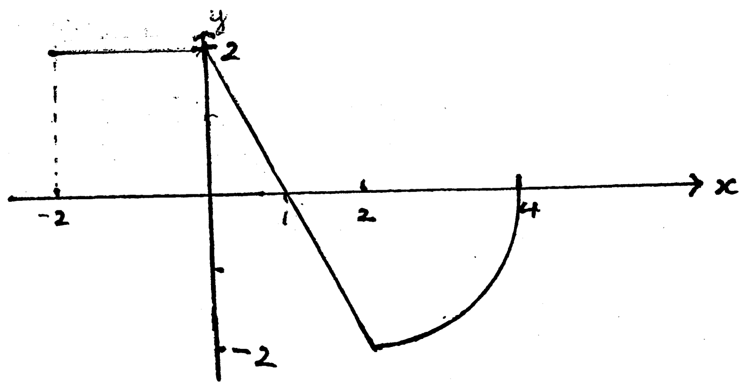
(i) What is the focal length?

(ii) What is the equation of the directrix?

(iii) What is the equation of the parabola?

b) Derive the equation of the locus of a point P that moves so that PA is twice the distance PB where A is $(0, 3)$ and B is $(4, 7)$. 3

c) Use area formulae to evaluate $\int_{-2}^{4} f(x)\, dx$ 2



4

d) Evaluate:

(i) $\int_1^3 (5-4x)\, dx$

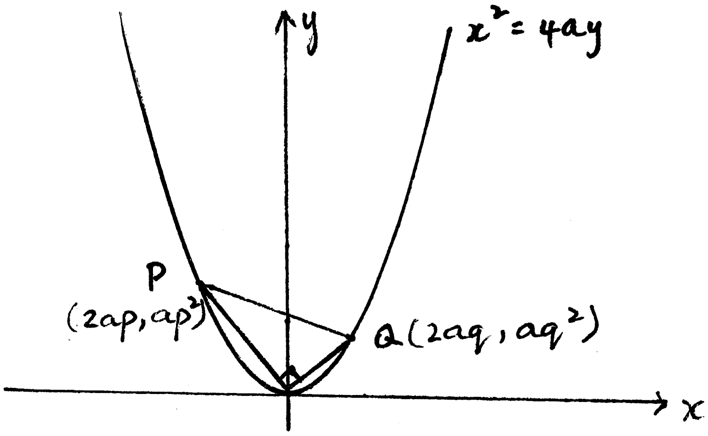
(iii) $\int_0^2 (2x-1)^3\, dx$

<u>Question 4</u> – (12 marks) – Start a new page **Marks**

a) For the function $y = f(x)$, $f(-1) = 3$ and $f(3) = 1$. Sketch the function from $x = -1$ to $x = 3$ if over this domain $f'(x) < 0$ and $f''(x) < 0$, and the function is continuous. 3

b) A curve contains the point $(1, 4)$ and its gradient is given by the function $(x+1)(x-2)$. Find the equation of the curve. 3

c)



The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two variable points on the parabola $x^2 = 4ay$. The chord PQ subtends a right angle at the origin. 6

(i) Find the gradient of OP

(ii) Show that $pq = -4$

M is the midpoint of PQ

(iii) Find the equation of the locus of M as P and Q vary.

(iv) Describe this locus.

Marks

<u>Question 5</u> – (12 marks) – Start a new page

a) (i) On the same axes sketch the graphs of $y = \sqrt{x}$ and $y = 6 - x$. 6

(ii) Show that $(4, 2)$ is the point of intersection of the two graphs.

(iii) Find the area bounded by the two graphs and the y-axis.

b) A rectangle is to be inscribed in the semicircle $y = \sqrt{4 - x^2}$ 6

(i) Show that the area of the rectangle can be given by $2x\sqrt{4 - x^2}$

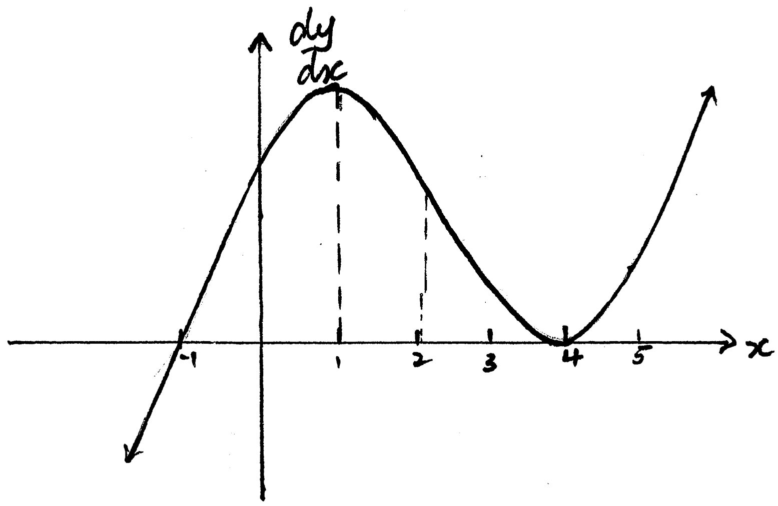
(ii) Find the dimensions of the rectangle with the greatest area.

<u>Question 6</u> – (12 marks) – Start a new page **Marks**

a) Show that $\int x(x+3)\,dx \neq \int x\,dx \int (x+3)\,dx$ 2

b) Find the volume of the solid formed when the region bounded by the parabola $y = 4 - x^2$ and the x-axis is rotated about the x-axis. 4

c)



Given the graph of the derivative of y, state values of the domain where the function y: 6

(i) is increasing

(ii) has a minimum turning point

(iii) has inflexional points

(iv) is concave up

End of Paper

YEAR 11 EXTENSION 1 Mark 4

ASSESSMENT TASK 1

DECEMBER 2007

QUESTION 1

a) $x^2 - 8x + y^2 + 14y + 1 = 0$

$x^2 - 8x + 16 + y^2 + 14y + 49 = -1 + 16 + 49$

$(x-4)^2 + (y+7)^2 = 64$

$= 8^2$

$\therefore$ THE CENTRE IS AT $(4,-7)$; THE RADIUS IS 8.

b) (i) $x^2 = 4ay$

$\therefore y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$

When $x = 2at$, $\frac{dy}{dx} = t$

Gradient of normal $= -\frac{1}{t}$

$\therefore$ Equation of normal is

$y - at^2 = -\frac{1}{t}(x - 2at)$

$ty - at^3 = -x + 2at$

$x + ty - at^3 - 2at = 0$

(ii) Let $x = 0$

$ty - at^3 - 2at = 0$

$y = at^2 + 2a$

$\therefore$ N is the point $(0, at^2 + 2a)$

c) $f(x) = 3^x$

x	0	1	2	3	4
3^x	1	3	9	27	81

$$\int_a^b f(x)\,dx \doteqdot \frac{b-a}{6}\left\{f(a) + 4f\left(\tfrac{a+b}{2}\right) + f(b)\right\}$$

$$\therefore \int_0^4 3^x\,dx \doteqdot \frac{2}{6}\left\{1 + 4\times3 + 2\times9 + 4\times27 + 81\right\}$$

$$= 73\tfrac{1}{3}.$$

d) (i) $\dfrac{d}{dx}\sqrt{1+x^4} = \dfrac{d}{dx}(1+x^4)^{\frac{1}{2}}$

$$= \tfrac{1}{2}(1+x^4)^{-\frac{1}{2}}.4x^3$$

$$= \frac{2x^3}{\sqrt{1+x^4}}$$

(ii) $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{2}\int \frac{2x^3}{\sqrt{1+x^4}}\,dx$

$$= \tfrac{1}{2}\sqrt{1+x^4} + C$$

QUESTION 2

a) $y = 4x^3 - x^4$

(i) $\frac{dy}{dx} = 12x^2 - 4x^3$

Let $\frac{dy}{dx} = 0$, $12x^2 - 4x^3 = 0$

$4x^2(3 - x) = 0$

$\therefore x = 0$ or $x = 3$

When $x = 0$, $y = 0$

$x = 3$, $y = 4 \times 27 - 81$

$= 27$

$\therefore$ There are 2 stationary points, $(0, 0)$ and $(3, 27)$

(ii) $\frac{d^2y}{dx^2} = 24x - 12x^2$

When $x = 3$, $\frac{d^2y}{dx^2} = 72 - 108$

< 0

$\therefore$ there is a maximum turning point at $(3, 27)$

When $x = 0$, $\frac{d^2y}{dx^2} = 0$

x	-1	0	1
$\frac{d^2y}{dx^2}$	-36	0	12

From the table, the concavity changes at $(0, 0)$

Hence there is a horizontal point of inflection at $(0, 0)$.

(iii) Let $\frac{d^2y}{dx^2} = 0$, $24x - 12x^2 = 0$

$12x(2 - x) = 0$

$x = 0$ or $x = 2$

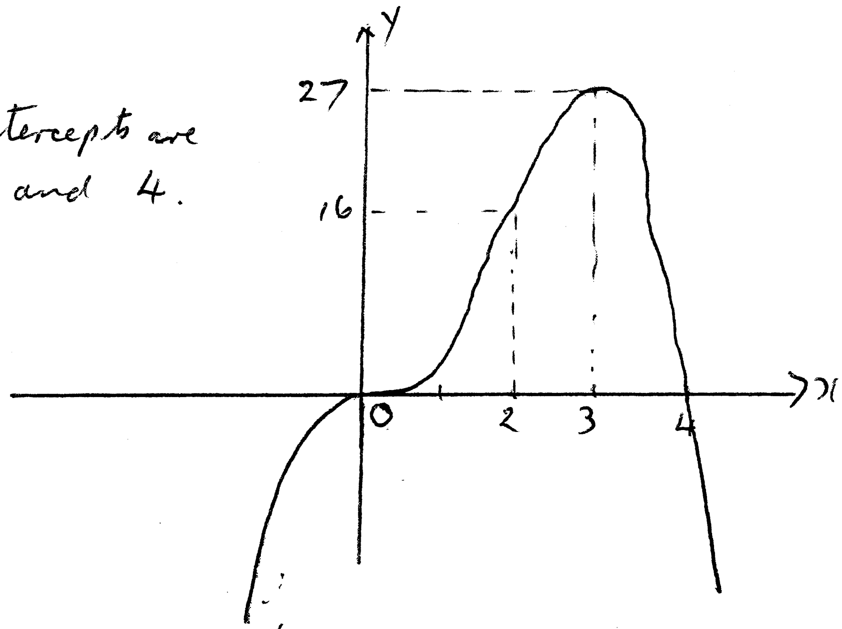
When $x = 2$, $y = 4 \times 8 - 16$

$= 16$

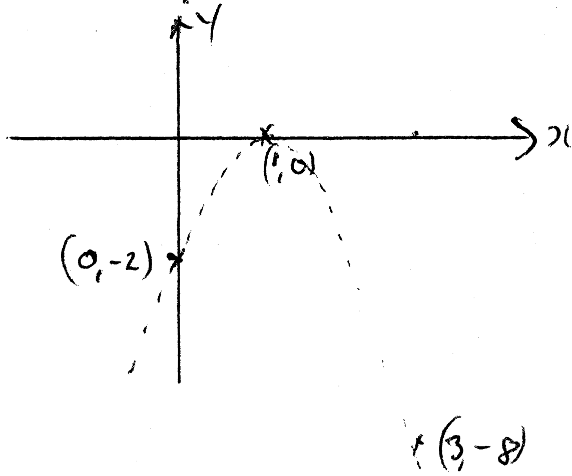
$\therefore$ Concavity changes at $(0, 0)$ and $(2, 16)$

(iv)

x intercepts are 0 and 4.



b)



$y = ax^2 + bx + c$.

Substitute $x = 0$, $y = -2$

$-2 = 0 + 0 + c$

$\therefore y = ax^2 + bx - 2$

Subst. $x = 1$, $y = 0$

$0 = a + b - 2$ (1)

Subst $x = 3$, $y = -8$

$-8 = 9a + 3b - 2$

$0 = 9a + 3b + 6$

$0 = 3a + b + 2$ (2)

(2) − (1)

$2a + 4 = 0$

$a = -2$

Subst in (1)

$b = 4$

$\therefore$ Equation is

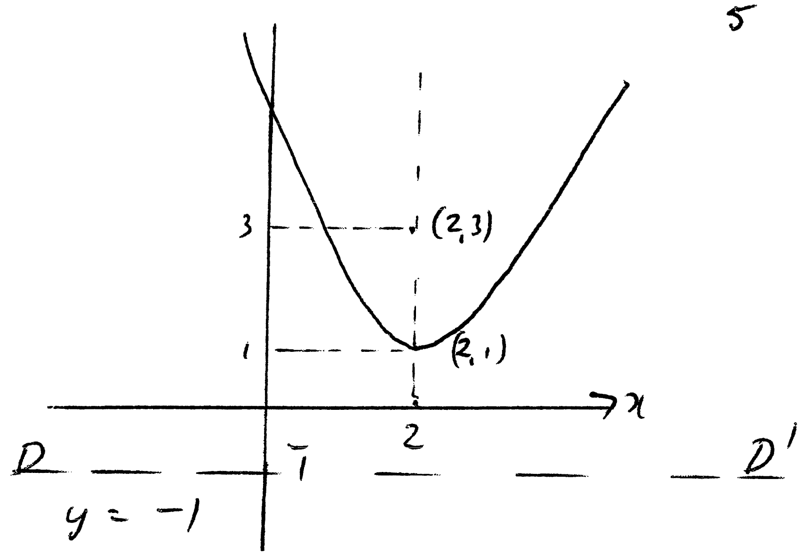
$y = -2x^2 + 4x - 2$

QUESTION 3

a)

(i) Focal length:

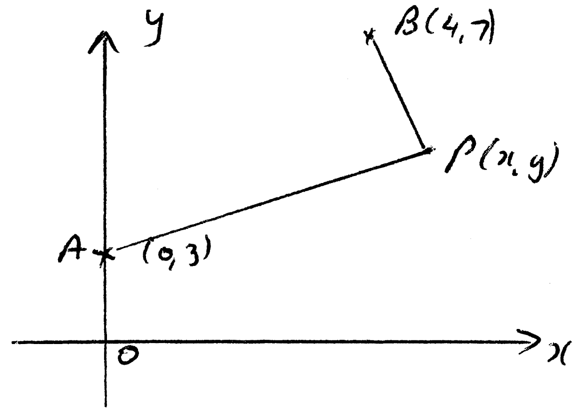
$a = 2$



(ii) Directrix is $y = -1$

(iii) Equation is $(x-2)^2 = 4 \times 2(y-1)$

$(x-2)^2 = 8(y-1)$



b) Let P be the point (x, y)

$PA = 2PB$

$PA^2 = 4PB^2$

$x^2 + (y-3)^2 = 4\left[(x-4)^2 + (y-7)^2\right]$

$x^2 + y^2 - 6y + 9 = 4\left[x^2 - 8x + 16 + y^2 - 14y + 49\right]$

$x^2 + y^2 - 6y + 9 = 4x^2 - 32x + 4y^2 - 56y + 260$

$\therefore 3x^2 + 3y^2 - 32x - 50y + 251 = 0$

c) $\int_{-2}^{4} f(x) = \frac{1}{2} \times 1 \times 2 - \frac{1}{2} \times 1 \times 2 - \frac{1}{4} \times \pi \times 2^2 + 4$

$= 4 - \pi$

d) (i) $\int_1^3 (5-4x)\,dx = \left[5x - 2x^2\right]_1^3$

$= (15-18) - (5-2)$

$= -6$

(ii) $\int_0^2 (2x-1)^3\,dx = \left[\frac{(2x-1)^4}{4\times 2}\right]_0^2$

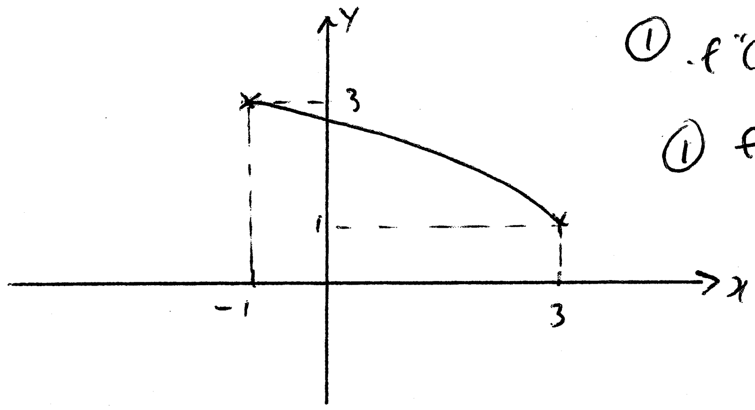
$= \frac{1}{8}\left(3^4 - (-1)^4\right)$

$= \frac{1}{8} \times 80$

$= 10$

QUESTION 4

a)



① . $f'(x) < 0$, decreasing

① . $f''(x) < 0$, concave down

① for scale, end pts.

b) $\frac{dy}{dx} = (x+1)(x-2)$ $\quad\frac{1}{2}$

$= x^2 - x - 2$

1

$\therefore y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + C$

Subst $x = 1$, $y = 4$ $\quad 1$

$4 = \frac{1}{3} - \frac{1}{2} - 2 + C$

$\therefore C = \frac{37}{6}$ $\quad\frac{1}{2}$

$\therefore y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{37}{6}$

$6y = 2x^3 - 3x^2 - 12x + 37$

c) (i) $M_{OP} = \frac{ap^2}{2ap}$ — (1)

$= \frac{p}{2}$

(ii) $M_{OQ} = \frac{q}{2}$

$PO \perp OQ$.

$\therefore \frac{p}{2} \cdot \frac{q}{2} = -1$ — (1)

$pq = -4$

(iii) MIDPOINT OF PQ IS $\left(\frac{2ap+2aq}{2}, \frac{ap^2+aq^2}{2}\right)$

$\left(a(p+q), \frac{a(p^2+q^2)}{2}\right)$ — (1)

$\therefore$ Parametric equations of M are

$x = a(p+q)$ (1)

$y = \frac{a(p^2+q^2)}{2}$

$y = \frac{a}{2}\left((p+q)^2 - 2pq\right)$ (2)

— (1)

From (1) $p+q = \frac{x}{a}$, and $pq = -4$

Sub in (2)

$y = \frac{a}{2}\left(\left(\frac{x}{a}\right)^2 - 2(-4)\right)$

$y = \frac{a}{2}\left(\frac{x^2}{a^2} + 8\right)$

$2y = \frac{x^2}{a} + 8a$ — (1)

$2ay = x^2 + 8a^2$

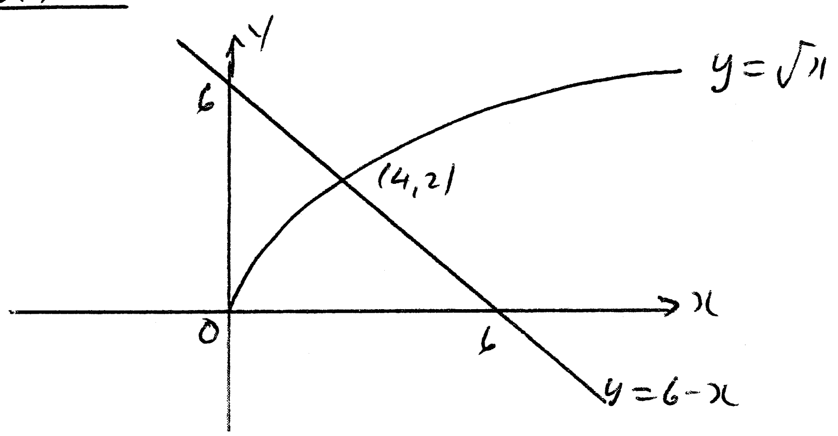
$\therefore x^2 = 2ay - 8a^2$

$x^2 = 2a(y - 4a)$?

(iv) $\therefore$ Locus is a parabola with vertex at $(0, 4a)$ with focal length $\frac{a}{2}$ — (1)

QUESTION 5

a) (i)



(ii) $y = \sqrt{x}$

$y = 6-x$

$\therefore \sqrt{x} = 6-x$

$x = (6-x)^2$

$= 36-12x+x^2$

i.e. $x^2-13x+36=0$

$(x-4)(x-9)=0$

$x=4$ or $x=9$

When $x=4$, $y = 6-4$

$= 2$

$\therefore$ (4,2) is the point of intersection of the 2 graphs

[Note: when $x=9$, $y=-3$ which is not on the graph $y=\sqrt{x}$.]

(iii) $A = \int_0^4 x^{\frac{1}{2}}\,dx + \int_4^6 6-x\,dx.$

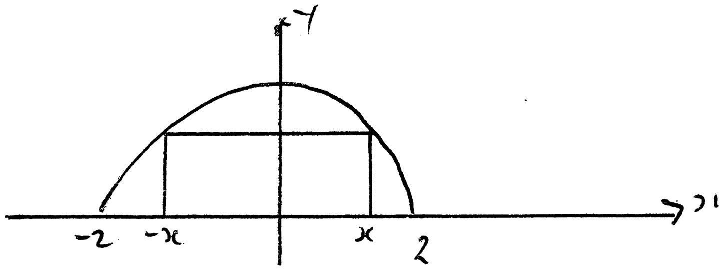
$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 + \left[6x-\frac{x^2}{2}\right]_4^6$

$= \frac{16}{3} - 0 + (36-18)-(24-8)$

$= \frac{16}{3} + 18 - 16$

$= 7\frac{1}{3}.$

b)



(i) Let one side of the rectangle lie on the x axis between $-x$ and x (NB symmetry)

$\therefore$ Length $= 2x$

Height $= \sqrt{4-x^2}$

$\therefore$ Area $= 2x\sqrt{4-x^2}$

(ii) Let $A = 2x(4-x^2)^{\frac{1}{2}}$

$$\frac{dA}{dx} = 2x \cdot \frac{1}{2}(4-x^2)^{-\frac{1}{2}} \cdot (-2x) + (4-x^2)^{\frac{1}{2}} \cdot 2$$

$$= -\frac{2x^2}{\sqrt{4-x^2}} + 2\sqrt{4-x^2}$$

$$= \frac{-2x^2 + 2(4-x^2)}{\sqrt{4-x^2}}$$

$$= \frac{8-4x^2}{\sqrt{4-x^2}}$$

Let $\frac{dA}{dx} = 0$ ie $\frac{8-4x^2}{\sqrt{4-x^2}} = 0$

$8-4x^2 = 0$

$2-x^2 = 0$

$x = \pm\sqrt{2}$.

x	1	$\sqrt{2}$	1.5
$\frac{dA}{dx}$	+	0	−

$\therefore$ Area will be a max when $x = \sqrt{2}$

When $x = \sqrt{2}$, $y = \sqrt{4-(\sqrt{2})^2}$

$= \sqrt{2}$

$\therefore$ Dimensions are $2\sqrt{2} \times 2$.

QUESTION 6

a) $$\int x(x+3)\,dx = \int x^2+3x\,dx$$

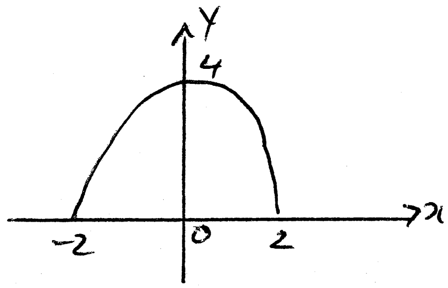
$$= \frac{x^3}{3}+\frac{3x^2}{2}+c$$

$$\int x\,dx . \int (x+3)\,dx = \frac{x^2}{2}\left(\frac{x^2}{2}+3x\right)+c$$

$$= \frac{x^4}{4}+\frac{3x^3}{2}+c$$

$$\neq \int x(x+3)\,dx.$$

b) $$V=\pi\int_a^b [f(x)]^2\,dx$$



$$= \pi\int_{-2}^{2}(4-x^2)^2\,dx$$

$$= 2\pi\int_0^2 16-8x^2+x^4\,dx$$

$$= 2\pi\left[16x-\frac{8x^3}{3}+\frac{x^5}{5}\right]_0^2$$

$$= 2\pi\left[32-\frac{64}{3}+\frac{32}{5}-0\right]$$

$$= \frac{512}{15}\pi$$

c) (i) $x>-1$, $x\neq 4$

(ii) $x=-1$

(iii) $x=1$, 4

(iv) $x<-1$